3d N=2 theories and plumbing graphs: matter, gauging and dualities

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based on 2302.13371 with Piotr Sułkowski

Outline

We will discuss Abelian theories with mixed Chern-Simons levels and matter

- **gauging and dualities**: SEQD-XYZ, mirror duality, ...
- **geometric construction**: three manifolds, brane webs, ...

3d Abelian theories

- For a theory with gauge group: $G = U(1) \times \cdots \times U(1)$, mixed Chern-Simons couplings come from

$$S_{CS} = k_{ij} \int A_i dA_j \,, \quad k_{ij} = k_{ji} \,.$$

  The effective Chern-Simons levels $k_{ij}^{\text{eff}} \in \mathbb{Z}$, because of chiral multiplets.
- Global symmetries:
  For each gauge group $U(1) \in G$, a topological symmetry $U(1)_T$, FI parameter ξ.
  For each Chiral multiplet, a flavor symmetry $U(1)_F$, mass parameter m.
- Superpotential: $\mathcal{W} = \Phi_1 \Phi_2 \Phi_3 + \cdots$.

Compactification

The DGG construction: tetrahedra $\to \Phi$, internal line $\to \mathcal{W} = \Phi_1\Phi_2\Phi_3 + \cdots$

- $6d$ $(2,0)$ theories (on M5-branes) $\xrightarrow{M_3}$ $3d$ $\mathcal{N}=2$ theories
- M_3 can be hyperbolic manifolds, plumbing manifold, and so on.

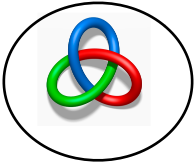
  [Gadde,Gukov,Putrov,2013][Gukov,Putrov,Vafa,2016][Gukov,Pei,Putrov,Vafa,2017]

The DGG's construction cannot easily construct theories with mixed CS levels.

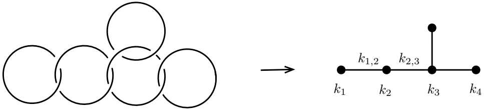
We hope directly use three-manifolds to construct 3d theories. Then the properties of three-manifolds play important roles: Dehn surgery, Kirby moves, ...

Three manifolds

A large class of three manifolds are constructed by Dehn surgery:



Three manifolds are represented by plumbing graphs with linking numbers K_{ij}



which engineer Abelian gauge theories:

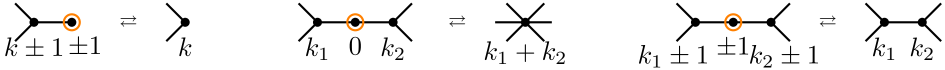
$$U(1) \times \cdots \times U(1), \text{ with mixed CS levels } K_{ij} \ .$$

However, there is no matter in this construction.

Kirby moves

There are equivalent graph descriptions of the same three manifold.

Kirby moves lead to equivalent theories



Kirby moves can be physically interpreted:

- Gauging the topological symmetry (add gauge nodes), from RHS to LHS
- Integrating out the gauge fields (reduce gauge nodes). from LHS to RHS

Gauging means adding a path integral for the background field to make it dynamic. $Z' = \int DAZ$. Gauging generates a new theory from the old one $\mathcal{T}' = \mathcal{T}/G$.

Integrating out gauge fields each gauge field of $U(1)_{k_i}$ can be separated:

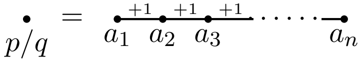
$$Z_{S_b^3} = \int dx\ e^{k_i x^2 + (\cdots)x} \times (\text{other terms})$$

Kirby moves can be applied to generic graphs:

,

An example: Lens space

Lens space: $L(p,q) = S^3_{p/q}$ leads to a theory $U(1)_{p/q}$ without matter, or a theory $U(1)_{a_1} \times U(1)_{a_2} \times \cdots \times U(1)_{a_n}$.



only if they satisfy a condition:

$$\frac{p}{q} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \cfrac{1}{\cdots - \cfrac{1}{a_n}}}} \qquad L(p,q) \longrightarrow ST^{a_1}ST^{a_2}S\cdots ST^{a_n}S\,.$$

Note that $S, T \in SL(2,\mathbb{Z})$, and S is interpreted as gauging the topological symmetry $U(1)_T$. Geometrically, S is a Dehn surgery between two lens spaces $S^3_{a_i}$ and $S^3_{a_j}$, gluing the torus of nearby lens spaces from 1-cycles (α,β) to (β,α).

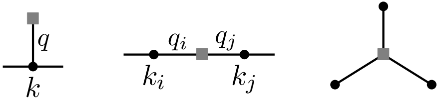
Questions

The three-manifold construction is still at the beginning.

- Could we add chiral multiplets to plumbing graphs?

Matter node

- We conjecture it is possible to add matter.
- A new notation: we denote the matter by ■ ← **matter node**
  Matters in fundamental, bifundamental, tri-fundamental representations are represented by graphs:



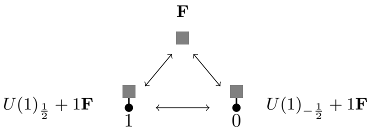
where q_i are charges under gauge nodes $U(1)_k$, $U(1)_{k_i} \times U(1)_{k_j}$, respectively.

We can use **duality** to understand plumbing graphs with matter.

Mirror triality

IItsy is a mirror duality between a free matter and a theory $U(1)_{\pm 1/2}$ with matter:

[Witten, 2003][Dimofte, Gaiotto, Gukov, 2011]



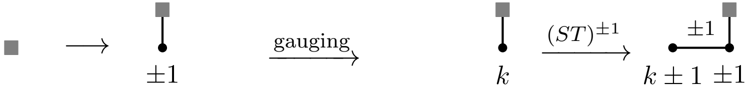
The 3d abelian theories enjoy a global $SL(2,\mathbb{Z})$ action. The nice property $(ST)^3 = 1$ leads to this mirror triality.

Their sphere partition functions are equivalent

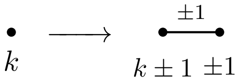
$$(ST)^{\pm} \ : \quad s_b\left(\frac{iQ}{2} - m\right) = e^{\mp \frac{\pi i}{2}\left(\frac{iQ}{2} - m\right)^2} \int dx\, e^{\mp \frac{\pi i}{2} x^2 \mp 2\pi i\, m x - \frac{\pi Q}{2} x} s_b\left(\frac{iQ}{2} - x\right) .$$

Gauging the 0-form symmetry

Gauging this mirror pair leads to an interesting move (we call it **ST-move**):



ST-move is analogous to **gauging** the top. sym. $U(1)_T$ (a Kirby move $TS^{\pm 1}T$):



What is the relation between ST-move and Kirby move?

$$\text{ST-move} \xrightarrow{\text{decouple the matter}} \text{Kirby move}$$

One can just remove the matter node in the large mass decoupling.

Note that the gauging the free matter leads to $1\mathbf{F}/U(1)_F = U(1)_{\pm 1/2} + 1\mathbf{F}$.

We can gauge the global symmetry $U(1)_F \leftrightarrow U(1)_T$ for the above mirror pair. After gauging, this symmetry becomes a new gauge symmetry.

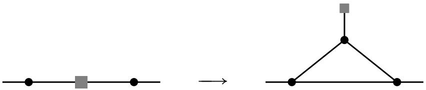
We can also turn on bare CS level and FI parameters for this new gauge symmetry. Note that after gauging, the mass parameter become dynamic, and the matter is massless.

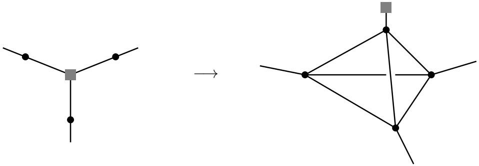
Matter in generic rep.

The background fields do no need to be unique. Gauging flavor symmetries $U(1)^2$ and $U(1)^3$ leads to bifundamental and tri-fundamental matter respectively.

The ST-move always turns a matter in generic representation with charges $(q_1, q_2, \cdots q_n)$ into a fundamental matter with charge 1.

Examples:

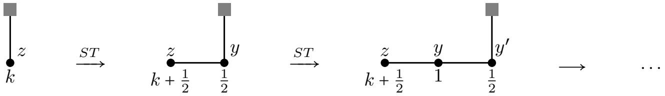


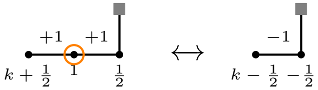


Hence, ST-moves significantly simplify plumbing graphs.

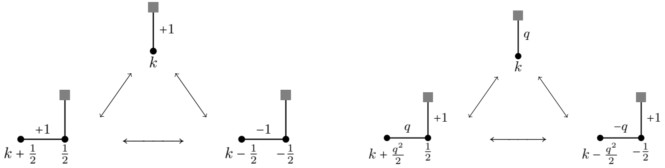
Gauged mirror triality

We can apply ST-moves many times, and there are actually two independent cases corresponding to $(ST)^{\pm 1}$ up to Kirby moves (integrating out gauge nodes).





These theories are dual to each other, namely $\mathcal{T} = \mathcal{T}/\{\text{ST-moves}, \text{Kirby moves}\}$:



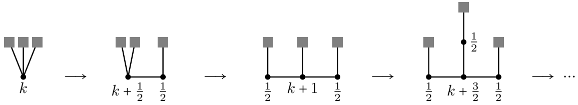
Equivalent operations

By now, we have discussed two equivalent operations on plumbing graphs.

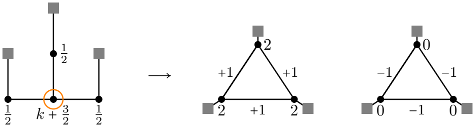
- gauge nodes $\longleftarrow$ Kirby moves
- matter nodes $\longleftarrow$ ST-moves

Examples: $U(1)_k + 3\mathbf{F}$

Kirby moves lead to many equivalent graphs



For $k = -\frac{5}{2}$ or $-\frac{1}{2}$, symmetric graphs

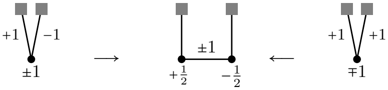


For $k = \pm\frac{3}{2}$, bifundamentals

Examples: mirror theories

Mirror dual theories have equivalent plumbing graphs. For example,

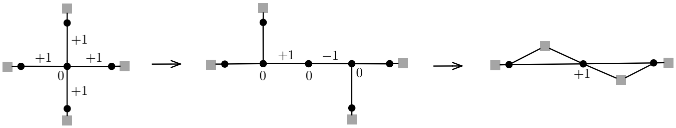
$$U(1)_{\pm 1} + 1\mathbf{F} + 1\mathbf{AF} \quad \xleftrightarrow{\text{mirror}} \quad U(1)_{\mp 1} + 2\mathbf{F}$$



The Abelian mirror pair [Dorey, Tong, 1999]

$$U(1)_{-\frac{N_f}{2}} + N_f\mathbf{F} \quad \xleftrightarrow{\text{mirror}} \quad [1] - U(1) - U(1) - \cdots - U(1) - [1]$$

is related through Kirby moves. For example,



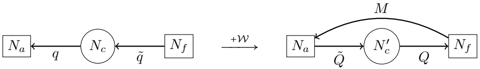
Questions

The above examples shows that plumbing graphs are very useful to prove mirror dualities, derive new dual theories, and analyze properties of theories.

Could plumbing graphs encode **superpotentials**?

Superpotential

Seiberg duality introduces a superpotential



One typical example of Seiberg-duality is the mirror dual pair ($\mathcal{T}_1 \leftrightarrow \mathcal{T}_2$):

$$1\mathbf{F} + 1\mathbf{AF} \quad \longleftrightarrow \quad U(1)_0 + 1\mathbf{F} + 1\mathbf{AF} + 1\mathbf{Adj} \;\; \text{with } \mathcal{W} = Q\tilde{Q}\Phi_{adj} \; ,$$

This duality comes from 3d $\mathcal{N}=4$ theory, and can be interpreted as S-duality (D5 $\leftrightarrow$ NS5, D3 $\leftrightarrow$ D3).

This duality can be equivalently viewed as the SEQD-XYZ duality:

$$U(1)_0 + 1\mathbf{F} + 1\mathbf{AF} \quad \longleftrightarrow \quad \{\mathbf{X}\ , \mathbf{Y}, \mathbf{Z}\} \quad \text{with } \mathcal{W} = \mathbf{XYZ}$$

Gauged SQED-XYZ

The flavor symmetry $U(1)_y \times U(1)_z$ that are associated with **F** and **AF** respectively.

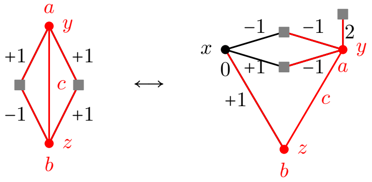
After gauging this symmetry, one can get a new dual pair. Their sphere partition functions

$$\int dy\, dz\, [\ldots]\; Z^{\mathcal{T}_1}_{S^3_b} = \int dy\, dz\, [\ldots]\; Z^{\mathcal{T}_2}_{S^3_b}$$

in which one can also turn on mixed CS levels (a, b, c) and FI parameters (ξ_y, ξ_z)

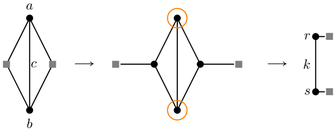
$$[\ldots] := e^{-\pi i (a y^2 + b z^2 + 2c\, yz)} e^{2\pi i\, (\xi_y y + \xi_z z)}\,,$$

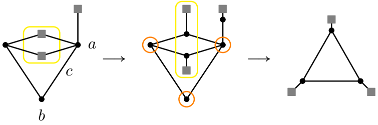
The corresponding dual plumbing graphs become



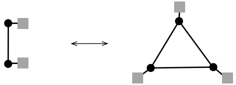
This dual pair looks complicated. One can use Kirby moves and ST-moves to simplify it.

This dual graph pair can be simplified by generic Kirby moves





The gauged SEQD-XYZ duality becomes a 2-3 move:



One can continue perform Kirby moves and ST-moves to generate many 2-3 moves.

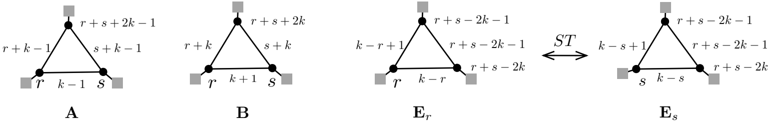
The mixed CS levels (linking numbers) should be integers, because of parity anomaly.

One can keep track of the transformations of sphere partition functions to read off the effective mixed CS levels.

$$\int dydz[...] \int dx\, e^{-2\pi i\, zx} s_b\left(-\frac{iQ}{2} + 2y\right) s_b\left(\frac{iQ}{2} \pm x - y\right) = \int dydz[...]\, s_b(y \pm z)$$

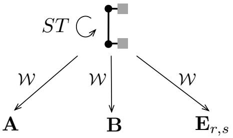
$$Z_{S_b^3} = \int dx_i e^{\frac{\widetilde{\mathcal{W}^{eff}}(k_{ij}^{\text{eff}},\xi)}{\hbar}+\mathcal{O}(\hbar)}$$

However, most of them are not integers, and only the following cases are meaningful:

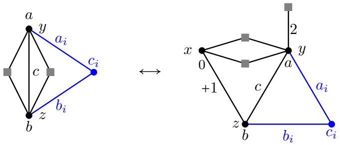


We call them superpotential triangles. When we gauge SQED-XYZ duality, we did not change superpotential. The sperpotential of the gauged XYZ model is still that of XYZ model $\mathcal{W} = XYZ$, which can be encoded implicitly in the relations of mixed CS levels of these triangles.

These are close relations between these triangles

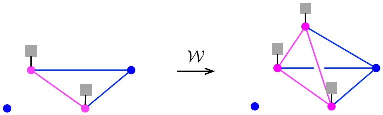


Gauged SEQD -XYZ duality is a Seiberg-like duality, which is a local duality of a big graphs. For instance, we can add external gauge nodes $U(1)_c$:



This external gauge node can be viewed as gauging the topological symmetry.

Once again, ST-move and Kirby move simply this generic case

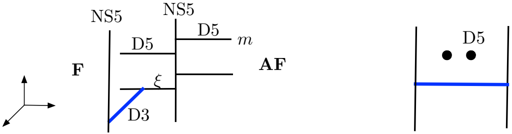


In the above, we fully gauge the flavor symmetry $U(1)_y \times U(1)_z$. We can also partially gauge the flavor symmetry. Namely, only gauge the $U(1)_y$ or $U(1)_z$, but this only leads to a subset of the results given by the fully gauging.

One can also consider the Higgsing of chiral multiplets using plumbing graphs.

Brane web

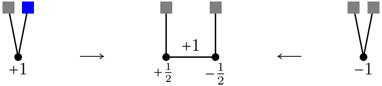
Various properties of plumbing graphs can be checked by 3d brane web.



For example, the dual pair can be interpreted as S-duality ($(p, q) \leftrightarrow (-q, p)$)

$$U(1)_{+1} + 1\mathbf{F} + 1\mathbf{AF} \quad \leftrightarrow \quad U(1)_{-1} + 2\mathbf{F}$$

ST-moves easily show their equivalence:



Similarly, we can show the equivalence of several theories from the perspective of both brane webs and plumbing graphs, such as

$U(1)_{\frac{1}{2}} + 1\mathbf{F} \quad \leftrightarrow \quad U(1)_{-\frac{1}{2}} + 1\mathbf{F}$

$U(1)_{\frac{3}{2}} + 1\mathbf{F} \quad \leftrightarrow \quad U(1)_{-\frac{3}{2}} + 1\mathbf{F}$

$U(1)_{\pm 1} + 1\mathbf{F} + 1\mathbf{AF} \quad \leftrightarrow \quad U(1)_{\mp 1} + 2\mathbf{F}$

$U(1)_{\mp\frac{1}{2}} + 2\mathbf{F} + 1\mathbf{AF} \quad \leftrightarrow \quad U(1)_{\pm\frac{1}{2}} + 1\mathbf{F} + 2\mathbf{AF}$

The last one needs gauged SQED-XYZ to prove this duality.

There are several operations on brane webs, such as S-duality $(p, q) \leftrightarrow (-q, p)$, reflection $(p, q) \leftrightarrow (p, -q)$, and movement of D5-brane. It is not easy to find the correspondence between these operations with ST-moves and Kirby moves on plumbing graphs.

However, the reflection $(p, q) \leftrightarrow (p, -q)$ of brane webs can be interpreted as reversing the orientation of plumbing manifolds (graphs) (bare CS levels $k_{ij} \leftrightarrow -k_{ij}$).

Summary and outlook

In the above, we get equivalent plumbing graphs by gauging some basics dualities. However, the geometric interpretation of these equivalent graphs is very incomplete. In particular, we should figure out how to add matter nodes to plumbing manifolds (work in progress with Satoshi Nawata).

Besides, the above story is only for abelian theories. It would be interesting to see if nonabelian theories also work well on plumbing graphs.

Thank you!