Plumbing Graphs with Matter

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based on a work (to appear) with P. Sułkowski

Motivation

- 3d $\mathcal{N} = 2$ theories have mixed Chern-Simons levels, which have not been extensively studied yet.
- Some pure abelian theories can be engineered by plumbing manifolds [Gukov, ...] .
- We want to discuss more abelian theories with generic matter.
- We want to understand knots-quivers correspondence [Sułkowski, ...] .

3d $\mathcal{N} = 2$ theories

- There are mixed Chern-Simons levels between gauge nodes $U(1) \times U(1)$

$$S_{CS} = k_{ij} \int A_i dA_j \,, \quad k_{ij} = k_{ji} \,,$$

  which receive contributions form chiral multiplets. The effective Chern-Simons levels should be integers.
- For each gauge node $U(1)$, there is a FI parameter ξ, which is associated to the topological symmetry $U(1)$.
- For each chiral multiplet, there is a real mass parameter m, which is associated to a flavor symmetry $U(1)$.
- There could be a superpotential $\mathcal{W}$ in the Lagrangian.

3d-3d correspondence

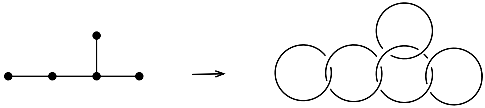
- Compactifying 6d $(2,0)$ theories on three-manifolds M_3 leads to 3d $N=2$ theories,

$$6d\ (2,0)\ \text{theories} \xrightarrow{M_3} 3d\ N=2\ \text{theories}\ T[M_3, G]$$

- We only consider abelian theories, and hence only one M5-brane wraps the M_3.
- In particular, the M_3 can be plumbing manifolds.

Plumbing graphs

- Plumbing manifolds are represented by plumbing graphs.
- Plumbing graphs: each black node denotes a gauge group $U(1)_k$. The lines are linking numbers.



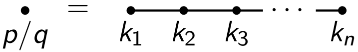
- Plumbing graphs encode the linking number K_{ij} between circles.

$$K_{ij} = \begin{cases} k_{ii}\,, & i = j \\ 1\,, & i \neq j \end{cases}$$

- Plumbing graphs engineers pure gauge theories $U(1) \times \cdots \times U(1)$ with mixed CS levels K_{ij}, if we ignore the decoupled adjoint chiral multiplets.

Plumbing graphs

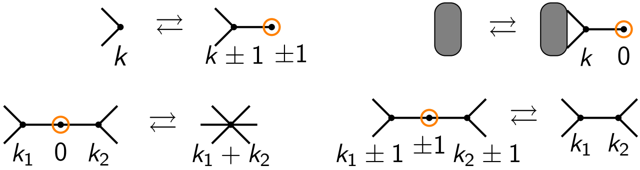
- For example, the lens space $L(p,q) = S^3_{-p/q}(\text{unknot})$ is denoted by



- Lens space gives rise to a pure gauge theory $U(1)_{p/q}$ or equivalently $U(1) \times \cdots \times U(1)$.
- This example suggests that there are many equivalent or dual plumbing theories, related by Kirby moves of gauge nodes.

Kirby moves for gauge nodes

- Kirby moves are some operations to produce equivalent graphs

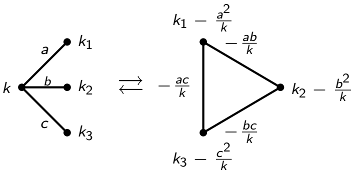


- Kirby moves can be interpreted as integrating out the gauge nodes (the node in orange circle).
- Because the contribution of $U(1)$ to sphere partition function is a Gaussian integral

$$Z_{S_b^3} = \int dx \; e^{kx^2 + (\cdots)x} \; \times (\text{other terms})\,.$$

Kirby moves for gauge nodes

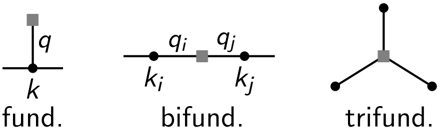
- For example,



- There is a problem left: there is no matter for plumbing graphs. We want to add chiral multiplet, such as fundamental matter **F**, anti-fundamental matter **AF**, bifundamental matter, and etc.

Matter

- One can add chiral multiplets to the plumbing theories.
- We use a gray box ■ to denote the matter with charge q under gauge nodes. For (anti)fundamental matter **F** and **AF**, $q = \pm 1$,



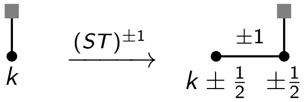
- After adding matter, plumbing graphs become some kinds of quiver diagrams.

Kirby moves for matter: ST-transformation

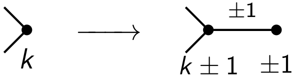
- For the Lagrangian of 3d theories, there is a $SL(2,\mathbb{Z})$ action [Witten]. $ST \in SL(2,\mathbb{Z})$ with $(ST)^3 = 1$.
- ST transformations change the polarization of M_3[Gukov] ,
- ST-transformation is the process of gauging a flavor symmetry and introduce a new gauge node

$$(ST)^{\pm}: \quad \mathcal{L}(A) \to \mathcal{L}(A) \pm \frac{1}{2}\tilde{A}dA \pm \frac{1}{4}\tilde{A}d\tilde{A}$$

- Graphically,



- Therefore, ST can be viewed as a Kirby move for matter, which is analogous to

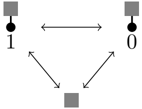


Kirby moves for matter: ST-transformation

- ST-transformation is a duality between theories, as its provides an mathematical identity for the contribution of matter

$$(ST)^{\pm} \ : \quad s_b\left(\frac{iQ}{2} - z\right) = e^{\mp\frac{\pi i}{2}\left(\frac{iQ}{2}-z\right)^2}\int dx\, e^{\mp\frac{\pi i}{2}x^2 \mp 2\pi i\, zx - \frac{\pi Q}{2}x} s_b\left(\frac{iQ}{2} - x\right)$$

- ST leads to the triality between $U(1)_{\pm 1/2} + 1\mathbf{F}$ and a free matter $\mathbf{F}$:

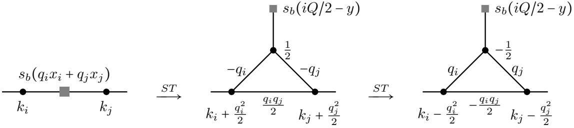


where linking numbers are effective CS levels.

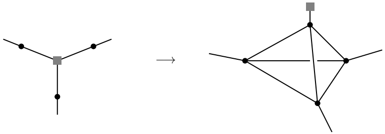
- ST can be viewed as a nontrival replacement that introduces new gauge nodes and a mixed CS level.

Kirby moves for matter: ST-transformation

- For bifundamental matter,



- For trifundamental matter,



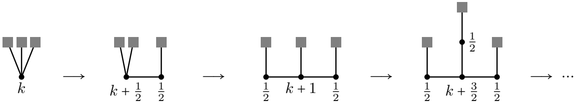
- Therefore, matter can be turned into **F** via ST.

Kirby moves for matter

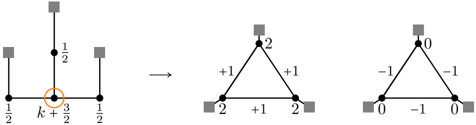
- One can Kirby move a matter for at most three time because of the equivalence relation $(ST)^3 = 1$.
- Kirby moves for gauge nodes and ST-transformation can be combined to generate equivalent plumbing graphs, which suggest dual theories, if we characterize 3d theories by mixed CS levels.

Examples: $U(1)_k + 3\mathbf{F}$

- One can perform Kirby moves



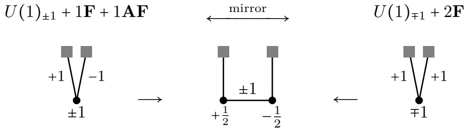
- For $k = -\frac{5}{2}$ or $-\frac{1}{2}$, one can get very symmetric graphs with loops



- For $k = \pm\frac{3}{2}$, the central node has a vanishing CS level $U(1)_0$, which leads to a Delta function $\delta(x_i)$ that reduces the number of gauge nodes. One can eventually get bifundamentals

Examples: mirror theories

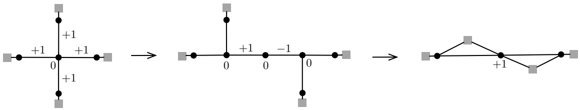
- Mirror dual theories have the same plumbing graphs.



- The Abelian mirror pair

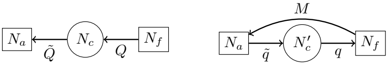
$$U(1)_{-\frac{N_f}{2}} + N_f \mathbf{F} \xleftrightarrow{\text{mirror}} [1] - U(1) - U(1) - \cdots - U(1) - [1]$$

can be related through Kirby moves. For example, when $N_f = 4$, one can inserted nodes $U(1)_0$



Seiberg-like duality

- Seiberg duality introduces a superpotential and a matter

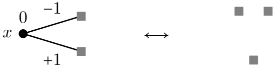


- One simple example is the SQED-XYZ dual pair

$$U(1)_0 + 1\mathbf{F} + 1\mathbf{AF} \quad \longleftrightarrow \quad \{X, Y, Z\} \text{ with } \mathcal{W} = XYZ$$

- The associated sphere partition functions are equivalent

$$\int dx e^{-2\pi i\, zx} s_b\left(iQ/2 \pm x - y\right) = s_b(y \pm z) s_b\left(iQ/2 - 2y\right)$$

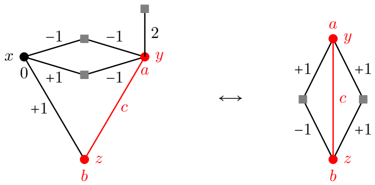


Seiberg-like duality

- However, Kirby moves for this mirror pair does lead to integrating theories.
- We should gauge the flavor symmetries for mass parameters and introduce FI parameters

$$\int \textbf{gauge} := \int dy\, dz\, e^{-\pi i (a y^2 + b z^2 + 2c\, yz)} e^{2\pi i (\xi_y y + \xi_z z)}$$

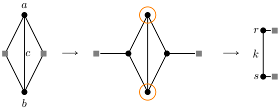
- The mirror pair becomes $\int \textbf{gauge}\ Z^{SQED} = \int \textbf{gauge}\ Z^{XYZ}$

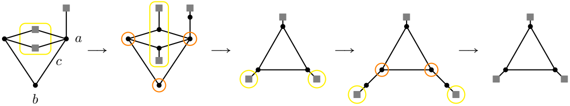


where we move one matter to the left graph.

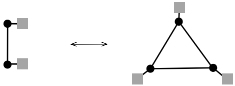
Seiberg-like duality

- Applying Kirby moves for the above graph pair leads to interesting theories.



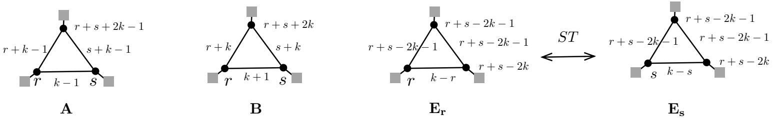


- The Seiberg-like dual graphs becomes 2-3 moves



Triangles A,B,E

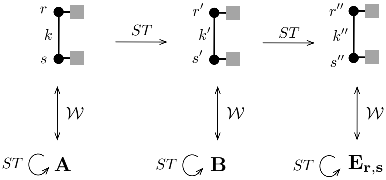
- The integral triangle obtained by Kirby moves is not unique.



- The cases **A** and **B** are firstly found in [Longi,], which are named unlinking and linking.

Triangles A,B,E

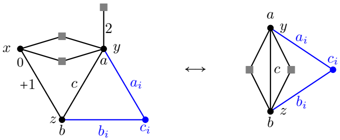
- The relations between these cases are



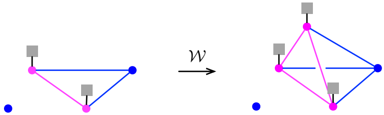
- Triangles **A**,**B** and **E** are related by an operation $\mathcal{W}^{-1} \circ (ST)^* \circ \mathcal{W}$.
- For specific r, k, s, there may be other triangles relating to these cases by ST-transformations.

With external nodes

- Seiberg dual graphs are local duality on subgraphs. In the presence of other nodes



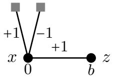
- Eventually, Kirby moves lead to the 2-3 move



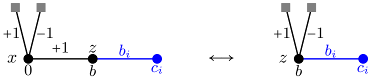
- If other nodes connect to the two pink nodes, then they also connect to the introduced node.

Partially gauging

- One can also partially gauge the flavor symmetries to decouple a matter, and becomes a 2-2 move



- If this dual pair couples to other gauge nodes



- Kirby moves on partial gauging graphs also fall in the cases **A**, **B** and **E**.

Higgsing

- If the mass parameters for **F** and **AF** are equal then Higgsing happens

$$1\mathbf{F} + 1\mathbf{AF} \to 1$$

- Which can be interpreted as Higging a D5-brane in 3d brane webs.

Outlook

- Construct nonabelian theories.
- More generic dualities, monopole superpotentials.
- Produce fundamental matter using M5-branes.
- Geometrically add matter.
- Other constructions, such as CY4, M2 probe orbifolds, quiver mutations, and 3d brane webs.
- Higher form symmetry.