Refined open topological strings: single brane

Shi Cheng

IFT. University of Warsaw

Joint work with Piotr Sułkowski

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outline

- Closed topological strings
  1. M-theory interpretation
  2. Refined topological vertex

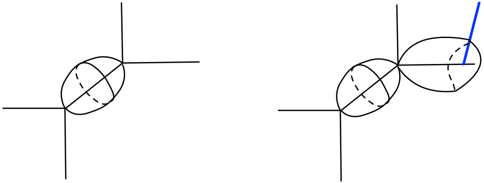
- Open topological strings
  1. Refined geometric transition
  2. $t, q, \bar{t}, \bar{q}$-branes

- Open BPS invariants
  1. Open GV formula
  2. Examples: strips, $\mathbb{F}_0$, $\mathbb{F}_2$.

- HW transitions

Topological strings in M-theory

- Topological strings have gauge theory interpretations in M-theory/IIA/IIB. In M-theory, M2-branes wrapping on spheres $\mathbb{P}^1$ are closed top. strings and give closed BPS states. M2-branes wrapping on discs $\mathbb{C}^1$ are open top. strings and give open BPS states.



closed top. string　　　　open top. string

- The summation of closed string contributions is closed partition function, and the summation of open string contributions is open partition functions.

Closed partition function

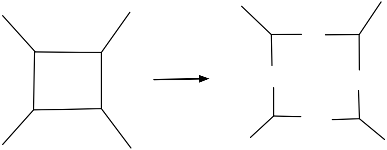
- Topological strings could be refined by the Lorentz symmetry $SU(2)_L \times SU(2)_R$ in spacetime of $\mathbb{R}^4 \times S^1 \times CY_3$.

$$\mathbb{R}^4 = \mathbb{C}_q \times \mathbb{C}_{\bar{t}}$$

$$(z_1, z_2) \to (qz_1, t^{-1}z_2)$$

- Refined closed partition functions can be calculated by refined topological vertex, which cuts toric diagrams into building blocks: vertex and edges. Then

$$Z^{\text{closed}}(Q, t, q) = \sum_{\mu} \prod (\text{Vertex factor}) \cdot \prod (\text{Edge factor})\,.$$



- Closed partition functions can be written in terms of Nekrasov factors.

$$Z^{closed} = Z^M \cdot \sum_{\mu} Q_i^{|\mu_\cdot|} ||Z_{\mu_\cdot}(t,q)||^2 \frac{\prod N_{\nu_\cdot}^{\mathrm{half},-}(Q_i)\ N_{\mu_\cdot \nu_\cdot}(Q_i)}{\prod N_{\mu_\cdot \nu_\cdot}(Q_i)} ,$$

where Nekrasov factors (functions) are defined as

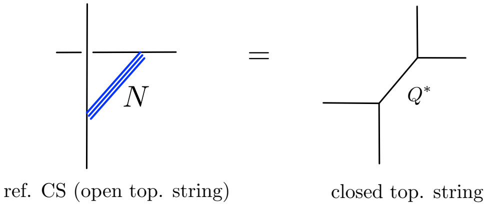
$$N_{\mu\nu}(Q;t,q) := \prod_{i,j=1}^{\infty} \frac{1 - Q\ q^{\nu_i - j}\ t^{\mu_j^T - i + 1}}{1 - Q\ q^{-j}\ t^{-i+1}} ,$$

$$N_{\nu}^{\mathrm{half},-}(Q) := N_{\nu\emptyset}\left(Q\sqrt{\frac{q}{t}}\right) ,\ N_{\nu}^{\mathrm{half},+}(Q) := N_{\emptyset\nu}\left(Q\sqrt{\frac{q}{t}}\right) .$$

- Nekrasov factors is powerful in gauge theory. As we will see later, Nekrasov factors determines geometric transitions.

Geometric transition

- Refined geometric transition is the open-closed duality in topological strings, which equals open topological strings to closed top. strings, and is the foundation for refined topological vertex. Taking conifold as an example



$$Z^{ref.CS}(t,q) = Z^{closed}(Q^*,t,q), \quad Q^* = t^N\sqrt{\frac{t}{q}} \text{ or } \frac{1}{q^N}\sqrt{\frac{t}{q}}$$

- N is the number of M5-branes on Lag-submfds. From ref. CS theory viewpoint, if N=0, NO brane, if N=1, Single t-brane or $\bar{q}$-brane in this case.

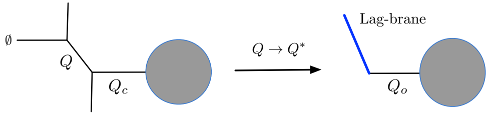
$t, q, \bar{t}, \bar{q}$-brane

- M5-brane on Lag $\times \mathbb{C}$ where $\mathbb{C}$ can be $\mathbb{C}_q$ or $\mathbb{C}_{\bar{t}}$, giving rise to q-brane and $\bar{t}$-brane respectively. Recall

$$\mathbb{R}^4 = \mathbb{C}_q \times \mathbb{C}_{\bar{t}}\,, \quad (z_1, z_2) \to (qz_1, t^{-1}z_2)\,.$$

- M2-branes ending on $q, \bar{q}, t, \bar{t}$-branes give rise to open BPS states.
- Geometric transition creates branes, and equals open top. strings to closed top. strings by tuning Q to some specific value Q^*.

$$\boxed{Z^{open} = Z^{closed}(Q^*)}$$



- Q^* is crucial. Going to ref CS theory is complicated for generic local CY_3. However, Q^* can be easily determined form half-Nekrasov factors $N_\nu^{\text{half},\pm}(Q^*)$ because of constraints

$$N_\nu^{\text{half},+}\left(q\sqrt{\frac{q}{t}}\right) \neq 0 \qquad \text{only if} \quad \nu = \{n\}$$

$$N_\nu^{\text{half},+}\left(\frac{1}{t}\sqrt{\frac{q}{t}}\right) \neq 0 \qquad \text{only if} \quad \nu = \{1,1,...,1\}$$

$$N_\nu^{\text{half},-}\left(t\sqrt{\frac{t}{q}}\right) \neq 0 \qquad \text{only if} \quad \nu = \{1,1,1,...,1\}$$

$$N_\nu^{\text{half},-}\left(\frac{1}{q}\sqrt{\frac{t}{q}}\right) \neq 0 \qquad \text{only if} \quad \nu = \{n\}$$

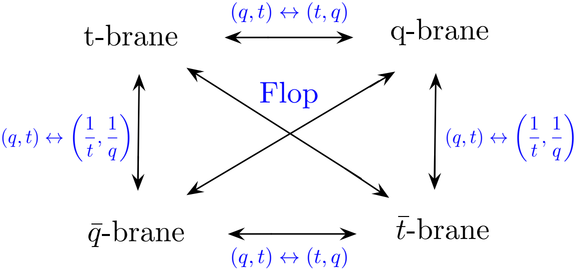
which constraint the Young diagram on single brane to be symmetric or anti-symmetric.

- Four types of branes can be defined by geometric transition

$$q\text{-brane}: \quad Q^* = q\sqrt{\frac{q}{t}}\,, \qquad t\text{-brane}: \quad Q^* = t\sqrt{\frac{t}{q}}\,,$$

$$\bar{q}\text{-brane}: \quad Q^* = \frac{1}{q}\sqrt{\frac{t}{q}}\,, \qquad \bar{t}\text{-brane}: \quad Q^* = \frac{1}{t}\sqrt{\frac{q}{t}}\,.$$

- These four types of branes are actually equivalent up to exchange symmetry, which are inherited from exchange symmetries of closed GV formula.

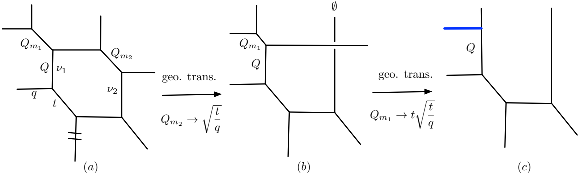


Open partition function

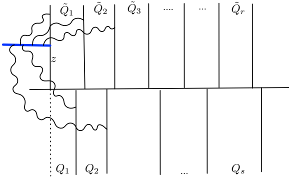
- Now, we already know open partition functions of $q, \bar{q}, t, \bar{t}$-brane can be obtained form closed partition functions by giving EVE to some Kähler parameter Q^*.
- Question: Could we use ref. top. vertex to calculate refined open partition function?
  Answer: YES, and NO
- Refined top. vertex was designed for toric diagrams, so it behaves unstable and may gives wrong result for non-toric diagrams. We need to be very careful.

Examples

- Geometric transition or Higgsing to create Lag-branes

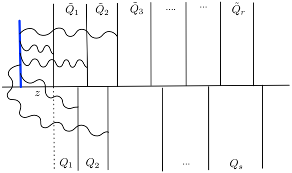


- Branes on strip geometry usually are hypergeometric functions, and encode infinity many open BPS states.



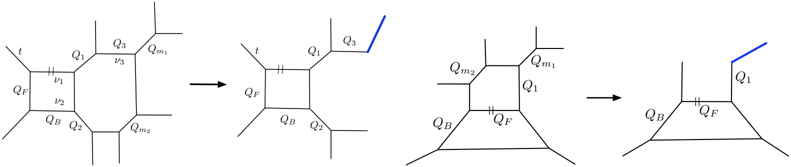
$$Z^{\text{open}}_{\bar{q}\text{-brane}}(z,\alpha_i,\beta_j) = \sum_{n=0}^{\infty} \frac{\left(z\sqrt{\frac{t}{q}}\right)^n}{(t,t)_n} \frac{\left(\alpha_1\sqrt{\frac{t}{q}},t\right)_n \left(\alpha_2\sqrt{\frac{t}{q}},t\right)_n \cdots \left(\alpha_s\sqrt{\frac{t}{q}},t\right)_n}{\left(\beta_1\frac{t}{q},t\right)_n \left(\beta_2\frac{t}{q},t\right)_n \cdots \left(\beta_r\frac{t}{q},t\right)_n},$$

- Branes associated finite many open BPS states.



$$Z^{\text{open}}_{t\text{-brane}}(z, \alpha_i, \beta_j) = \frac{(z\beta_1\sqrt{qt}, q)_\infty (z\beta_2\sqrt{qt}, q)_\infty \cdots (z\beta_r\sqrt{qt}, q)_\infty}{(zt, q)_\infty (z\alpha_1 t, q)_\infty (z\alpha_2 t, q)_\infty \cdots (z\alpha_s t, q)_\infty} .$$

- Straightforwardly, geometric transition gives open top. strings on local CY with compact divisors.



- How to count open BPS states?

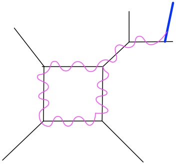
Open Gopakuma-Vafa formula

- Closed topological strings satisfy Gopakuma-Vafa formula. Open topological strings should also similar formula. For t-brane, $Z_{\mathrm{BPS}}^{\mathrm{open}} = e^{\mathcal{F}_{t\text{-brane}}}$,

$$\mathcal{F}_{t\text{-brane}} = \sum_{\beta \in H_2(X,L,\mathbb{Z})} \sum_{s,r} \sum_{n=1}^{\infty} \frac{(-1)^{2s} N_{\beta}^{(s,r)} q^{-ns} t^{n\left(r+\frac{1}{2}\right)}}{n\left(q^{n/2} - q^{-n/2}\right)} Q_{\beta}^{n}$$

where (s, r) should be the combination of spin and r-charge of symmetry $SO(2)_{\mathbb{C}_t} \times SO(2)_{\mathrm{R}}$, and (s, r) can be negative. $N_{\beta}^{s,r}$ are called open BPS invariants. $N_{\beta}^{s,r}$ is the number of BPS states in the same representation. One can also write down open GV formulas for other branes using exchange symmetry. $N_{\beta}^{s,r}$ are the same for different types of branes.

- Open strings could wrap around the boundary of compact divisor.



Q1 Q3 QB^4 QF^3

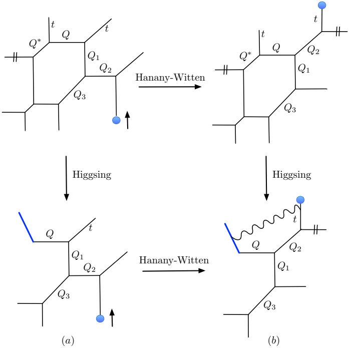
{1, 0, 1, 4, 3}

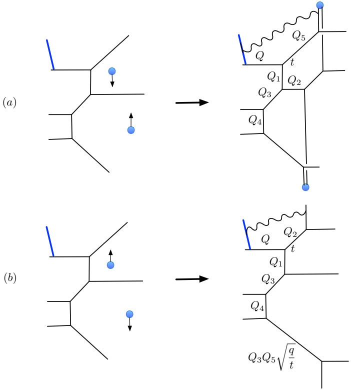
2r\2s	-23	-21	-19	-17	-15	-13	-11	-9	-7	-5	-3	-1	1	3	5	7	9	11	13	15	17	19	21	23	25
-23							1																		
-21						1	2	1																	
-19					1	3	6	3	1																
-17				1	3	8	12	9	3	1															
-15			1	3	9	16	24	19	10	3	1														
-13		1	3	8	16	28	38	37	22	10	3	1													
-11	1	2	6	12	24	38	55	57	45	23	10	3	1												
-9		1	3	9	19	37	57	77	70	48	23	10	3	1											
-7			1	3	10	22	45	70	93	77	49	23	10	3	1										
-5				1	3	10	23	48	77	102	80	49	23	10	3	1									
-3					1	3	10	23	49	80	107	81	49	23	10	3	1								
-1						1	3	10	23	49	81	109	81	49	23	10	3	1							
1							1	3	10	23	49	81	110	81	49	23	10	3	1						
3								1	3	10	23	49	81	109	81	49	23	10	3	1					
5									1	3	10	23	49	81	107	80	49	23	10	3	1				
7										1	3	10	23	49	80	102	77	48	23	10	3	1			
9											1	3	10	23	49	77	93	70	45	22	10	3	1		
11												1	3	10	23	48	70	77	57	37	19	9	3	1	
13													1	3	10	23	45	57	55	38	24	12	6	2	1
15														1	3	10	22	37	38	28	16	8	3	1	
17															1	3	10	19	24	16	9	3	1		
19																1	3	9	12	8	3	1			
21																	1	3	6	3	1				
23																		1	2	1					
25																			1						

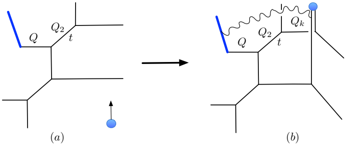
Hanany-Witten transition

- Through geometric engineering, closed topological strings engineer 5d $\mathcal{N} = 1$ SCFTs, while open topological strings engineer 3d-5d coupled gauge theories.
- Toric diagrams in M-theory dual to brane webs of 5d $\mathcal{N} = 1$ SCFTs in IIB. HW: One can move 7-branes freely without changing BPS spectrum by creating extra 5-branes when crossing 5-branes.
- Q: Does HW transition change open topological strings?
  A: Almost not, but creating some extra open strings.

- The open partition functions of following brane webs are equivalent if throwing away some open strings created by HW transition.







Conclusion and outlook

- Conclusion: Open topological strings are different from closed topological strings. Many operations, such as HW transitions and Flops, do not change closed strings but change open strings.
- Refined top. vertex does not work well for non-toric diagrams (CY with brane can be regarded as some kind of non-toric diagrams), we need to use other method, such as VOA construction of top. vertex to ignores non-toric structure.
- Multiple brane cases and branes on internal lines are subtle problems. Many open strings emerge and connect among branes, even if branes are top of each other.
- Modular transformations of open partition functions to relate different placements of branes.
- Refined open BPS wall crossing.
- Does quiver representation for open top. strings works beyond strip geometry? Does quiver equal to toric geometry? What is the CY condition for quivers?

Thank you very much.